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DYNAMIC REPRESENTATION FORMULAS AND FUNDAMENTAL SOLUTIONS FOR PIEZOELECTRICITY

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Abstract Representation formulas and fundamental solutions of the governing equations of transient piezoelectricity are obtained through a generalization of the reciprocal theorem and the plane wave transform method. It is shown that dynamic fundamental and singular solutions can be reduced to one-dimensional integral expressions by means of a slowness surface. The article provides the necessary mathematical foundations towards the development of the boundary element method for nonstationary, three-dimensional electroelastic problems.

1. INTRODUCTION

Problems involving deformable anisotropic dielectrics pose, in general, formidable mathematical complexities. A typical example is furnished by piezoelectric materials. In such a case, strong electromechanical coupling effects preclude closed form solutions except for cases dealing with bodies of infinite extension subjected to simple loading conditions. Piezoelectric ceramics and polymers are finding a growing number of important applications in aerospace, automotive, medical and electronic technologies. It is not surprising, therefore, that new efforts are being concentrated on the search for numerical procedures as a means to predict the behavior of these materials under severe loading conditions.

Because of its versatility, efficiency and economy of numerical calculations, the boundary element method (BEM) appears to be a good alternative to treat piezoelectric boundary-value problems. As a consequence, recent articles have been devoted to two important aspects of the BEM formulation: (a) the construction of fundamental solutions, i.e. Green's functions; (b) the derivation of representation formulas. The latest examples are provided by the works of Lee and Jiang (1994), Chen and Lin (1993) and Chen (1993). A common deficiency, however, in these and other articles dealing with numerical approaches is that the formulations are restricted to time-independent behavior. Applications of piezoelectric materials in the areas of electromechanical devices and electronic packaging are but two examples to illustrate the fact that the transient response of the material is an important phenomenon and, hence, cannot be neglected. Therefore, it becomes clear that more general and physically realistic formulations are a basic need to completely characterize the mechanical and electrical behavior of piezoelectric-based structures. Within this general framework, Norris (1994) has discussed the derivation of dynamic Green's functions for problems dealing with piezoelectricity and other coupling phenomena in anisotropic media. His elegant paper, however, is constrained to time harmonic equations and, furthermore, the final expressions for the fundamental solutions are given in a transformed domain only.

While the present article relies upon concepts also found in Norris (1994), it has no restrictions regarding the behavior of the equations with respect to time and, as it will be shown, the final results are presented in fairly simple forms and quite suitable for numerical implementation. Such features are of extreme importance in view of the fact that the ultimate goal is the development of a general purpose BEM computer program for applications in the area of electrodynamics of deformable bodies. Towards this end, representation formulas are derived by making use of a generalization of the reciprocal theorem of elastodynamics which includes electrical effects. These representation formulas are given in terms of dynamic fundamental solutions to the equations of transient piezoelectricity and their derivatives. The main body of the paper is, in fact, devoted to the derivation of explicit expressions for such fundamental and singular solutions. The mathematical approach is based on the plane wave transform, which consists of representing the three-dimensional Dirac-delta function by means of an integral over the unit sphere (Gelfand and Shilov, 1964).

An important feature of our paper consists of showing that the aforementioned unit sphere representation is just one manner of describing the fundamental solutions. In fact, simpler expressions can be obtained by means of alternative representations. In a paper devoted to particular cases of aelotropic elastodynamics, Burridge (1967) has shown that the Green's functions can also be represented by integrals over the so-called slowness surface. In the present paper we generalize his work to transform the Green's functions originally derived over the unit sphere into integrals evaluated over the slowness surface of an arbitrary piezoelectric body. These results, however, are only an intermediate step towards the derivation of expressions that constitute one of the most unique aspects of the present article. Indeed, we show that the dynamic fundamental and singular solutions of piezoelectricity can be represented by one-dimensional integrals along a path determined by the intersection of the slowness surface with a plane moving in the direction of the field vector (Khutoryansky, 1985). It is important to note that, despite the complexities involved in dynamic piezoelectric problems, the one-dimensional representations are strikingly simple in form.

As a prelude to the numerical implementation of the representation formulas and fundamental solutions in a BEM program, we provide a discussion on the behavior of the solutions in the neighborhood of the body's boundary, as well as the nature of the singularities involved in the representation formulas. Moreover, we derive by means of the corresponding static solutions a special regularized form of the representation formula, which is continuous across the boundary and, more importantly, it also holds at boundary points where the normal does not exit. We conclude the article with a brief discussion regarding the nature of the boundary conditions most typically encountered in electroelastic problems.

2. MATHEMATICAL PRELIMINARIES

We make use of both direct and component (or indicial) notation within the framework of Cartesian coordinates. In the former case, tensors of rank one or above and their matrix representations are denoted by bold face letters. In the case of component notation we invoke the summation convention over repeated Latin subindices, which can be of two types with different ranges: $i, j, k, l = 1, 2, 3$ and $M, N = 1, 2, 3, 4$. Moreover, partial differentiation with respect to a space variable is denoted with a comma, while the time derivative is indicated with a dot over the variable or, alternatively, by the symbol ∂_t.

We consider a homogeneous piezoelectric body $\mathscr{B}$ with boundary $\partial\mathscr{B}$ whose motion in the Euclidean space is described in terms of the independent variables $\mathbf{x} = \{x_i\}$ and t. The description of the body is done through the equations of motion and Gauss' law, namely

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{f} = \rho\ddot{\mathbf{u}}$$
$$\operatorname{div}\mathbf{D} = q, \tag{1a,b}$$

where $\boldsymbol{\sigma} = \{\sigma_{ij}\}$, $\mathbf{u} = \{u_i\}$, $\mathbf{D} = \{D_i\}$, ρ, $\mathbf{f} = \{f_i\}$ and q denote stress, elastic displacement, electric displacement, mass density, body force per unit of volume and electric charge density, respectively. In addition, let $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}$ and $\mathbf{E} = \{E_i\}$ be the strain and electric field, respectively, given by

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})$$
$$\mathbf{E} = -\nabla\phi, \tag{2}$$

where the second equation, with ϕ being the electric potential, is a consequence of the quasi-electrostatic approximation.

The set of equations (1) and (2) is complemented with constitutive relations derived formally from appropriate thermodynamic potentials. There are four different ways of expressing these relations. If the strain and the electric field are chosen as the independent variables, the thermodynamic potential is the electric enthalpy $\Psi(\boldsymbol{\varepsilon}, \mathbf{E})$ (Parton and Kudryavtsev, 1988), given by

$$\Psi = U(\boldsymbol{\varepsilon}, \mathbf{D}) - \mathbf{E} \cdot \mathbf{D}, \tag{3}$$

where U is the internal energy expressed in terms of strain and electric displacement. As a result, it can be shown that the constitutive relations for linear piezoelectricity become

$$\begin{aligned} \sigma_{ij} &= \frac{\partial \Psi}{\varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k \\ D_i &= -\frac{\partial \Psi}{E_i} = e_{ikl}\varepsilon_{kl} + \epsilon_{ik}E_k, \end{aligned} \tag{4}$$

where C_{ijkl}, e_{ijk} and ϵ_{ij} are the elastic (measured at constant electric field), piezoelectric and dielectric (measured at constant strain) material constants, respectively, satisfying the following symmetry relations:

$$C_{ijkl} = C_{ijlk} = C_{jikl} = C_{klij}; \quad e_{kij} = e_{kji}; \quad \epsilon_{ik} = \epsilon_{ki}.$$

The combination of eqns (1) and (2) with (4) results in a system of four partial differential equations coupling the displacement's components and electric potential, namely

$$\begin{aligned} C_{ijkl}u_{k,lj} + e_{kij}\phi_{,kj} + f_i &= \rho \ddot{u}_i \\ e_{ikl}u_{k,li} - \epsilon_{ik}\phi_{,ki} &= q. \end{aligned} \tag{5a,b}$$

Notice that in eqn (5b) there is no time rate due to the quasi-electrostatic approximation. This means that the time in ϕ appears as a parameter, and time dependence is induced only by the displacement $\mathbf{u}$.

The solution of eqn (5) is constrained to mechanical and electrical boundary conditions, which can be written as

$$\begin{aligned} \mathbf{u} &= \bar{\mathbf{u}} \quad \text{on} \quad \partial\mathscr{B}_u \\ \boldsymbol{\sigma}\boldsymbol{\nu} &= \bar{\mathbf{t}} \quad \text{on} \quad \partial\mathscr{B}_t \\ [\![\phi]\!] &= 0 \quad \text{on} \quad \partial\mathscr{B}_\phi \\ [\![\mathbf{D}]\!]\cdot\boldsymbol{\nu} &= \bar{w} \quad \text{on} \quad \partial\mathscr{B}_w, \end{aligned} \tag{6a–d}$$

where $\bar{\mathbf{u}}$, $\bar{\mathbf{t}}$ and $\bar{w}$ are the values of displacement, traction and surface charge, respectively, prescribed over different portions of the boundary $\partial\mathscr{B}$ (or interfaces separating different materials), whose outward unit normal is denoted by $\boldsymbol{\nu} = \{\nu_i\}$. Moreover, $[\![f]\!] = f^+ - f^-$ represents the jump of the enclosed quantity across the boundary or interface. Finally, initial conditions must be specified for the elastic displacement and its first derivative, i.e.

$$\begin{aligned} \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}) \\ \dot{\mathbf{u}}(\mathbf{x}, 0) &= \mathbf{v}_0(\mathbf{x}). \end{aligned} \tag{7}$$

A few words regarding the initial conditions are in order. In the sequel we consider the values of σ, $\mathbf{u}$, $\mathbf{D}$, ϕ, $\mathbf{f}$ and q to be defined for $t \in (-\infty, \infty)$ and to be equal to zero if $t < 0$. In such a case it is clear that eqn (1b) is valid for all values of t. Moreover, if the values of $\mathbf{u}_0$ and $\mathbf{v}_0$ are equal to zero, then eqn (1a) is also valid for all values of t. However, if the values of $\mathbf{u}_0$ and $\mathbf{v}_0$ do not vanish, we have to change the definitions of the body force in order to include initial conditions in the equations of motion. In such a case the body force is given by (Gelfand and Shilov, 1964)

$$\mathbf{f}_0(\mathbf{x}, t) = H(t)\mathbf{f}(\mathbf{x}, t) + \rho\mathbf{u}_0(\mathbf{x})\dot{\delta}(t) + \rho\mathbf{v}_0(\mathbf{x})\delta(t), \tag{8}$$

where $\delta(t)$ is the Dirac-delta function and $H(t)$ is the unit step function, i.e. $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t \geqslant 0$. As a result, the equation of motion becomes

$$\operatorname{div}\sigma + \mathbf{f}_0(\mathbf{x}, t) = \rho\ddot{\mathbf{u}}(\mathbf{x}, t). \tag{9}$$

Although in the rest of the paper we use $\mathbf{f}$ as a matter of simplification in notation, the developments apply to both zero and nonzero initial conditions, where in the latter case we simply need to replace $\mathbf{f}$ by $\mathbf{f}_0$ according to eqn (8).

3. REPRESENTATION FORMULAS

Consider two electroelastic states, namely $[\mathbf{u}, \sigma, \phi, \mathbf{D}]$ and $[\tilde{\mathbf{u}}, \tilde{\sigma}, \tilde{\phi}, \tilde{\mathbf{D}}]$. The first state represents the solution to piezoelectric problems with finite domains and general loading conditions. The second state is of artificial nature and represents the fundamental solution to the case of an infinite piezoelectric medium subjected to an impulsive point force and an impulsive point charge. Each state is assumed to satisfy the governing equations introduced in the previous section. More specifically:

$$\text{First state} \begin{cases} \sigma_{ij,j} + f_i = \rho\ddot{u}_i \\ D_{i,i} = q \\ \sigma_{ij} = C_{ijkl}u_{k,l} + e_{kij}\phi_{,k} \\ D_i = e_{ikl}u_{k,l} - \epsilon_{ik}\phi_{,k} \end{cases} \tag{10}$$

$$\text{Second state} \begin{cases} \tilde{\sigma}_{ij,j} + \tilde{f}_i = \rho\ddot{\tilde{u}}_i \\ \tilde{D}_{i,i} = \tilde{q} \\ \tilde{\sigma}_{ij} = C_{ijkl}\tilde{u}_{k,l} + e_{kij}\tilde{\phi}_{,k} \\ \tilde{D}_i = e_{ikl}\tilde{u}_{k,l} - \epsilon_{ik}\phi_{,k}, \end{cases} \tag{11}$$

where the body force $\tilde{\mathbf{f}}$ and the electric charge density $\tilde{q}$ for the second state will be specified at a later stage. Let $\mathbf{t} = \sigma\mathbf{v}$ and $D_v = \mathbf{D}\cdot\mathbf{v}$ be the traction and normal components of the electric displacement, respectively, at the boundary. Furthermore, let a function Π_{12} be defined as follows:

$$\Pi_{12} = \int_{\mathscr{B}} f_i * \tilde{u}_i \,\mathrm{d}V + \int_{\partial\mathscr{B}} t_i * \tilde{u}_i \,\mathrm{d}A - \int_{\mathscr{B}} q * \tilde{\phi}\,\mathrm{d}V + \int_{\partial\mathscr{B}} D_v * \tilde{\phi}\,\mathrm{d}A, \tag{12}$$

where the first two terms represent mechanical work and the last two terms represent electrical work with signs in accordance with the use of the electric enthalpy (3). The symbol $*$ denotes the operation of convolution in time, which for arbitrary functions f and g is defined as

$$f*g = \int_{-\infty}^{t} f(\mathbf{x}, t-\tau) g(\mathbf{x}, \tau)\,\mathrm{d}\tau. \tag{13}$$

Likewise, let a second function Π_{21} be given by

$$\Pi_{21} = \int_{\mathscr{B}} \tilde{f}_i * u_i \,\mathrm{d}V + \int_{\partial\mathscr{B}} \tilde{t}_i * u_i \,\mathrm{d}A - \int_{\mathscr{B}} \tilde{q} * \phi \,\mathrm{d}V + \int_{\partial\mathscr{B}} \tilde{D}_v * \phi \,\mathrm{d}A, \tag{14}$$

where $\tilde{\mathbf{t}} = \tilde{\boldsymbol{\sigma}}\boldsymbol{v}$ and $\tilde{D}_v = \tilde{\mathbf{D}} \cdot \boldsymbol{v}$ on $\partial\mathscr{B}$. However, it can be shown that eqns (12) and (14) are equal to each other. The proof is accomplished by recognizing that

$$\int_{\partial\mathscr{B}} [t_i * \tilde{u}_i + D_v * \tilde{\phi}] \,\mathrm{d}A = \int_{\mathscr{B}} [-f_i * \tilde{u}_i + \rho \ddot{u}_i * \tilde{u}_i + \sigma_{ij} * \tilde{u}_{i,j}] \,\mathrm{d}V + \int_{\mathscr{B}} [q * \tilde{\phi} + D_i * \tilde{\phi}_{,i}] \,\mathrm{d}V,$$

as a consequence of the divergence theorem and the field equations. Inserting this relation into eqn (12) yields

$$\Pi_{12} = \int_{\mathscr{B}} [\sigma_{ij} * \tilde{u}_{i,j} + \rho \ddot{u}_i * \tilde{u}_i] \,\mathrm{d}V + \int_{\mathscr{B}} D_i * \tilde{\phi}_{,i} \,\mathrm{d}V. \tag{15}$$

Furthermore, for linear piezoelectricity the following relation holds true:

$$\sigma_{ij} * \tilde{u}_{i,j} + D_i * \tilde{\phi}_{,i} = \tilde{\sigma}_{ij} * u_{i,j} + \tilde{D}_i * \phi_{,i}$$

and since

$$\rho \ddot{u}_i * \tilde{u}_i = \rho \ddot{\tilde{u}}_i * u_i,$$

it follows that

$$\Pi_{12} = \Pi_{21}. \tag{16}$$

Equation (16) is the starting point towards the derivation of representation formulas for electroelasticity. Such a derivation is based on two independent loading conditions for the second (artificial) state, where a unit force and a unit charge (viewed as special distributions of body forces and charge density) are applied at a point $\xi = (\xi_1, \xi_2, \xi_3)$ of the piezoelectric medium (commonly known as the "source point"). We make a clear distinction between these two loads by treating them separately as follows:

(I) Let the body force and electric charge density for the second state be given by

$$\tilde{\mathbf{f}}(\mathbf{x}, t) = \delta(\mathbf{x} - \xi)\delta(t)\mathbf{e}_j, \quad \tilde{q}(\mathbf{x}, t) = 0, \tag{17}$$

where $\mathbf{e}_j$ is a unit vector along the x_j-axis, specifying the direction of the unit force, and $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$. We seek a solution to the system (11) in terms of functions $\tilde{\mathbf{u}}(\mathbf{x}, t)$ and $\tilde{\phi}(\mathbf{x}, t)$ satisfying the causality principle, i.e.

$$\tilde{\mathbf{u}}(\mathbf{x}, t) = \mathbf{0}, \quad \tilde{\phi}(\mathbf{x}, t) = 0 \quad \text{if } t < 0. \tag{18}$$

When the applied load is given by eqns (17) it is useful to introduce the following definitions:

$$\tilde{u}_i(\mathbf{x}, t) = U_{ij}(\mathbf{x}, \xi, t)$$
$$\tilde{\phi}(\mathbf{x}, t) = U_{4j}(\mathbf{x}, \xi, t)$$

$$\tilde{t}_i(\mathbf{x}, t) = T_{ji}(\xi, \mathbf{x}, t)$$
$$\tilde{D}_v(\mathbf{x}, t) = T_{j4}(\xi, \mathbf{x}, t), \tag{19}$$

where U_{ij} and U_{4j}, commonly known in the literature as Green's functions, represent the displacement (in the i-direction) and the electric potential, respectively, at the field point $\mathbf{x}$ due to a unit force applied at ξ in the j-direction. Likewise, T_{ji} and T_{j4}, which are the derivatives of the Green's functions, represent the traction on the boundary (in the i-direction) and the normal component of the electric displacement, respectively, at $\mathbf{x}$ when the previous unit force is applied at ξ. The use of a different order in subindices and arguments for the functions U_{ij} and T_{ji} is not arbitrary and will become more meaningful at a later stage of this section. Using eqns (17)–(19) in eqn (16) yields

$$\begin{aligned} u_j(\xi, t) = & \int_{\partial\mathscr{B}} \int_{0^-}^{t} [U_{ij}(\mathbf{x}, \xi, t-\tau)t_i(\mathbf{x}, \tau) + U_{4j}(\mathbf{x}, \xi, t-\tau)D_v(\mathbf{x}, \tau)]\, \mathrm{d}\tau\, \mathrm{d}A(\mathbf{x}) \\ & - \int_{\partial\mathscr{B}} \int_{0}^{t} [T_{ji}(\xi, \mathbf{x}, t-\tau)u_i(\mathbf{x}, \tau) + T_{j4}(\xi, \mathbf{x}, t-\tau)\phi(\mathbf{x}, \tau)]\, \mathrm{d}\tau\, \mathrm{d}A(\mathbf{x}) \\ & + \int_{\mathscr{B}} \int_{0^-}^{t} [U_{ij}(\mathbf{x}, \xi, t-\tau)f_i(\mathbf{x}, \tau) - U_{4j}(\mathbf{x}, \xi, t-\tau)q(\mathbf{x}, \tau)]\, \mathrm{d}\tau\, \mathrm{d}V(\mathbf{x}), \quad \xi \in \mathscr{B}, \end{aligned} \tag{20}$$

which is the representation formula for the elastic displacement. Equality (20) is a generalization of the representation formula of elastodynamics, where in this case electrical effects are also taken into consideration.

(II) Let the body force and electric charge density for the second state be given by

$$\tilde{\mathbf{f}}(\mathbf{x}, t) = \mathbf{0}, \quad \tilde{q}(\mathbf{x}, t) = -\delta(\mathbf{x}-\xi)\delta(t). \tag{21}$$

Again, the solution to eqns (11) is assumed to satisfy the causality principle (18) and a new set of Green's functions and their derivatives is introduced, according to

$$\tilde{u}_i(\mathbf{x}, t) = U_{i4}(\mathbf{x}, \xi, t)$$
$$\tilde{\phi}(\mathbf{x}, t) = U_{44}(\mathbf{x}, \xi, t)$$
$$\tilde{t}_i(\mathbf{x}, t) = T_{4i}(\xi, \mathbf{x}, t)$$
$$\tilde{D}_v(\mathbf{x}, t) = T_{44}(\xi, \mathbf{x}, t), \tag{22}$$

where the meaning of the variables on the right-hand side is similar to that of eqn (19), except that in this case a negative unit charge is applied at the source point. Using eqns (18) and (22) in eqn (16) yields the representation formula for the electric potential, namely

$$\begin{aligned} \phi(\xi, t) = & \int_{\partial\mathscr{B}} \int_{0}^{t} [U_{i4}(\mathbf{x}, \xi, t-\tau)t_i(\mathbf{x}, \tau) + U_{44}(\mathbf{x}, \xi, t-\tau)D_v(\mathbf{x}, \tau)]\, \mathrm{d}\tau\, \mathrm{d}A(\mathbf{x}) \\ & - \int_{\partial\mathscr{B}} \int_{0}^{t} [T_{4i}(\xi, \mathbf{x}, t-\tau)u_i(\mathbf{x}, \tau) + T_{44}(\xi, \mathbf{x}, t-\tau)\phi(\mathbf{x}, \tau)]\, \mathrm{d}\tau\, \mathrm{d}A(\mathbf{x}) \\ & + \int_{\mathscr{B}} \int_{0}^{t} [U_{i4}(\mathbf{x}, \xi, t-\tau)f_i(\mathbf{x}, \tau) - U_{44}(\mathbf{x}, \xi, t-\tau)q(\mathbf{x}, \tau)]\, \mathrm{d}\tau\, \mathrm{d}V(\mathbf{x}), \quad \xi \in \mathscr{B}. \end{aligned} \tag{23}$$

Formulas (20) and (23) can be combined into a single expression by introducing the matrices

$$\mathbf{U} = \begin{bmatrix} U_{ij} & U_{i4} \\ U_{4j} & U_{44} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_{ij} & T_{i4} \\ T_{4j} & T_{44} \end{bmatrix} \tag{24}$$

and the four-dimensional vectors

$$\mathbf{d} = \begin{Bmatrix} \mathbf{u} \\ \phi \end{Bmatrix}, \quad \mathbf{p} = \begin{Bmatrix} \mathbf{t} \\ D_n \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} \mathbf{f} \\ -q \end{Bmatrix}. \tag{25}$$

It should be clear that, fully expanded, $\mathbf{U}$ and $\mathbf{T}$ are 4×4 matrices and that in component form the four-dimensional vectors are denoted by d_N, p_N and b_N. Thus, when $N = 1, 2, 3$ the vectors are associated with mechanical variables, while when $N = 4$ they relate to an electrical variable. Using eqns (24) and (25), the representation formulas can be written in matrix form as

$$\mathbf{d}(\xi, t) = \int_{\partial\mathscr{B}}\int_0^t \mathbf{U}^{\mathrm{T}}(\mathbf{x}, \xi, t-\tau)\mathbf{p}(\mathbf{x}, \tau)\,\mathrm{d}\tau\,\mathrm{d}A(\mathbf{x}) - \int_{\partial\mathscr{B}}\int_0^t \mathbf{T}(\xi, \mathbf{x}, t-\tau)\mathbf{d}(\mathbf{x}, \tau)\,\mathrm{d}\tau\,\mathrm{d}A(\mathbf{x})$$
$$+\int_{\mathscr{B}}\int_0^t \mathbf{U}^{\mathrm{T}}(\mathbf{x}, \xi, t-\tau)\mathbf{b}(\mathbf{x}, \tau)\,\mathrm{d}\tau\,\mathrm{d}V(\mathbf{x}), \quad \xi\in\mathscr{B}. \tag{26}$$

A further simplification in notation follows by noting that if the body occupies the whole Euclidean space we have that

$$\mathbf{U}(\xi, \mathbf{x}, t) = \mathbf{U}^{\mathrm{T}}(\mathbf{x}, \xi, t), \tag{27}$$

where the superscript "T" denotes the transpose of the matrix. This equality can be obtained by using eqn (26) at time $\tau = 0$ and by replacing $\mathbf{b}(\mathbf{x}, t)$ by an impulsive load $\mathbf{f}$ or by a concentrated charge q. As a consequence of this symmetry condition, the matrix representation formula† becomes

$$\mathbf{d}(\xi, t) = \int_{\partial\mathscr{B}}\int_0^t \mathbf{U}(\xi, \mathbf{x}, t-\tau)\mathbf{p}(\mathbf{x}, \tau)\,\mathrm{d}\tau\,\mathrm{d}A(\mathbf{x}) - \int_{\partial\mathscr{B}}\int_0^t \mathbf{T}(\xi, \mathbf{x}, t-\tau)\mathbf{d}(\mathbf{x}, \tau)\,\mathrm{d}\tau\,\mathrm{d}A(\mathbf{x})$$
$$+\int_{\mathscr{B}}\int_0^t \mathbf{U}(\xi, \mathbf{x}, t-\tau)\mathbf{b}(\mathbf{x}, \tau)\,\mathrm{d}\tau\,\mathrm{d}V(\mathbf{x}), \quad \xi\in\mathscr{B}. \tag{28}$$

That is, the operation of transposition is eliminated and all arguments of $\mathbf{U}$ and $\mathbf{T}$ appear in the same order.

Let us recall that representation formulas are an important component of a BEM. As its name suggests, this numerical methodology relies upon integrations made on the boundary of the body. Therefore, a complete knowledge of the behavior of the functions involved in the first two integrals of eqn (28) is required. The following sections are devoted to the evaluation of the fundamental solutions $\mathbf{U}$ and their derivatives $\mathbf{T}$ and also to the study of their behavior on $\partial\mathscr{B}$.

4. FUNDAMENTAL SOLUTIONS

It is clear that to evaluate the integrals involved in eqn (28) we must first know the matrices $\mathbf{U}(\xi, \mathbf{x}, t)$ and $\mathbf{T}(\xi, \mathbf{x}, t)$. The latter, however, can be obtained from the former through the constitutive relations, as we show in Section 6. In the present section we limit ourselves to the derivation of the components of $\mathbf{U}$. We start by noting that because the body is assumed to be homogeneous we can write

†In the remainder of the article we prefer to use the wording "representation formula" when addressing eqn (28), although it is clear that in reality there are four formulas involved.

$$U_{MN}(\xi, \mathbf{x}, t) = U_{MN}(0, \mathbf{x}-\xi, t) = U_{MN}(\mathbf{x}-\xi, t), \tag{29}$$

where the first equality is a result of a shift of coordinate axes and in the last parenthesis we omit the zero for purposes of simplification. Moreover, if we assume the load to be placed at the origin of coordinates, we can write $U_{MN}(\mathbf{x}, t)$ instead of $U_{MN}(\mathbf{x}-\xi, t)$. We will follow this simplification throughout most of the article. Next, let $\mathbf{L}\,(\nabla, \partial_t)$ be a differential operator given by

$$\mathbf{L}(\nabla, \partial_t) = \begin{Vmatrix} \rho\partial_t^2 & 0 \\ 0 & 0 \end{Vmatrix} - \begin{Vmatrix} \mathbf{A} & \mathbf{a} \\ \mathbf{a}^{\mathrm{T}} & -\alpha \end{Vmatrix}, \tag{30}$$

where $\mathbf{A}(\nabla)$, $\mathbf{a}(\nabla)$ and $\alpha(\nabla)$ are tensors of rank two, one and zero, respectively, with components

$$A_{ik} = C_{ijkl}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}, \quad a_i = e_{kil}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}, \quad \alpha = \epsilon_{ik}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}. \tag{31}$$

As a consequence, eqns (5) can be written with direct notation as

$$\mathbf{L}(\nabla, \partial_t)\mathbf{d} = \mathbf{b}. \tag{32}$$

Furthermore, when the loads in the right-hand side of eqn (32) are given by the point loads (17) and (21), the above relation becomes

$$\mathbf{L}(\nabla, \partial_t)\mathbf{U}(\mathbf{x}, t) = \delta(t)\delta(\mathbf{x})I, \tag{33}$$

where $\mathbf{I}$ is the 4×4 unit matrix. In order to solve eqn (33) we make use of the plane wave transform. That is, $\delta(\mathbf{x})$ and $\mathbf{U}$ can be represented by integrals over the unit sphere $|\mathbf{n}| = 1$ in the following manner (Gelfand and Shilov, 1964):

$$\delta(\mathbf{x}) = -\frac{1}{8\pi^2}\nabla^2\int_{|\mathbf{n}|=1}\delta(\omega)\,\mathrm{d}\Omega(\mathbf{n}) \tag{34}$$

$$\mathbf{U}(\mathbf{x}, t) = \int_{|\mathbf{n}|=1}\mathbf{V}(\mathbf{n}, \omega, t)\,\mathrm{d}\Omega(\mathbf{n}), \quad \omega = \mathbf{n}\cdot\mathbf{x}, \tag{35}$$

where $\mathrm{d}\Omega(\mathbf{n})$ is the unit sphere's surface element, ∇^2 is the Laplacian operator and $\mathbf{V}(\mathbf{n}, \omega, t)$ is a new function which relates to $\mathbf{U}$ through eqn (35). In passing, we note that in the literature of generalized functions, such a relationship receives the name of Radon transform. Using eqns (34) and (35) in (33) yields the differential equation for the plane wave $\mathbf{V}(\mathbf{n}, \omega, t)$, namely

$$\mathbf{L}\left(\mathbf{n}\frac{\partial}{\partial\omega}, \frac{\partial}{\partial t}\right)\mathbf{V}(\mathbf{n}, \omega, t) = -\frac{1}{8\pi^2}\delta(t)\delta''(\omega)\mathbf{I}, \tag{36}$$

where the prime denotes the derivative with respect to ω. Since $\mathbf{U}$ and $\mathbf{V}$ are proportional, the latter also has the 4×4 matrix representation shown in eqns (24). Moreover, for a fixed column, $\mathbf{V}$ can be represented by a four-dimensional column vector $\{\mathbf{v}, \psi\}^{\mathrm{T}}$, where $\mathbf{v}$ is a three-dimensional vector, physically related to the components of the displacement vector associated to a fixed direction of the point force (for the first three columns of $\mathbf{V}$) or to the point charge (for the last column of $\mathbf{V}$). On the other hand, ψ is a scalar related to the electric potential and associated with a point force or point charge depending on the column under consideration. Furthermore, letting $\mathbf{i}_M$ be a unit vector with components $\{\delta_{1M}, \delta_{2M}, \delta_{3M}\}$, eqn (36) can be written as

$$\rho\ddot{\mathbf{v}} - \mathbf{A}\mathbf{v}'' - \mathbf{a}\psi'' = -\frac{1}{8\pi^2}\delta(t)\delta''(\omega)\mathbf{i}_M$$

$$-\mathbf{a}\cdot\mathbf{v}'' + \alpha\psi'' = -\frac{1}{8\pi^2}\delta(t)\delta''(\omega)\delta_{4M}, \tag{37a,b}$$

where it must be kept in mind that $\mathbf{A}$, $\mathbf{a}$ and α are now functions of the direction vector $\mathbf{n}$. Using eqn (37b) to solve for ψ'' and taking into consideration that $\alpha(\mathbf{n})$ is positive definite, we obtain

$$\psi'' = \frac{\mathbf{a}\cdot\mathbf{v}''}{\alpha} - \frac{1}{8\pi^2}\frac{\delta(t)\delta''(\omega)}{\alpha}\delta_{4M}, \tag{38}$$

which once inserted in eqn (37a) produces

$$\rho\ddot{\mathbf{v}} - \mathbf{B}\mathbf{v}'' = -\frac{1}{8\pi^2}\delta(t)\delta''(\omega)\mathbf{F}, \tag{39}$$

where

$$\mathbf{F}(\mathbf{n}) = \mathbf{i}_M + \frac{\mathbf{a}}{\alpha}\delta_{4M}$$

$$\mathbf{B}(\mathbf{n}) = \mathbf{A} - \frac{\mathbf{a}\otimes\mathbf{a}}{\alpha}. \tag{40}$$

That is, the original system of coupled partial differential equations given by (33) has been reduced to a differential equation for the variable $\mathbf{v}$, which by the causality principle can be expressed in terms of a new function as follows:

$$\mathbf{v}(\mathbf{n},\omega,t) = H(t)\mathbf{w}(\mathbf{n},\omega,t), \tag{41}$$

where $\mathbf{w}(\mathbf{n}, \omega, t)$ is C^1 in $[0, \infty)$ as a function of time. By virtue of the properties of the delta function, the derivatives of eqn (41) yield

$$\ddot{\mathbf{v}}(\mathbf{n},\omega,t) = \dot{\delta}(t)\mathbf{w}(\mathbf{n},\omega,0) + \delta(t)\dot{\mathbf{w}}(\mathbf{n},\omega,0) + H(t)\ddot{\mathbf{w}}(\mathbf{n},\omega,t)$$

$$\mathbf{v}''(\mathbf{n},\omega,t) = H(t)\mathbf{w}''(\mathbf{n},\omega,t), \tag{42}$$

which once substituted in eqn (39) give

$$\rho\ddot{\mathbf{w}} - \mathbf{B}\mathbf{w}'' = 0, \quad t > 0, \tag{43}$$

together with the initial conditions

$$\mathbf{w}(\mathbf{n},\omega,0) = \mathbf{0}$$

$$\dot{\mathbf{w}}(\mathbf{n},\omega,0) = -\frac{1}{8\pi^2\rho}\delta''(\omega)\mathbf{F}(\mathbf{n}). \tag{44}$$

Since eqn (43) has constant coefficients and all derivatives are of the same order, it follows that $\mathbf{w}$ has plane wave solutions of arbitrary shape. Hence, we consider a trial solution of the form (John, 1955)

$$\mathbf{w} = \mathbf{c}g(\omega - \lambda t), \tag{45}$$

where $\mathbf{c}$ is a constant vector and $g(\cdot)$ is a scalar valued function with nonzero second derivatives. More precisely, eqn (45) represents a plane wave with polarization vector $\mathbf{c}$ and wave front normal $\mathbf{n}$, propagating with velocity λ parallel to $\mathbf{n}$. Substituting eqn (45) into (43) reduces the differential equation to a system of three algebraic linear equations, namely

$$[\rho\lambda^2\mathbf{1} - \mathbf{B}]\mathbf{c} = \mathbf{0}, \tag{46}$$

where $\mathbf{1}$ is the 3×3 unit matrix. Nontrivial solutions of eqn (46) exist if $\mu = \rho\lambda^2$ is an eigenvalue of $\mathbf{B}$. Since the linear operator $\mathbf{B}$ is bounded, symmetric and positive definite (Balakirev and Gilinskii, 1982), its three eigenvalues (denoted by μ_1, μ_2 and μ_3) are real positive numbers, uniformly upper and lower bounded with respect to the vector $\mathbf{n}$. Hence,

$$\lambda_i = \sqrt{\frac{\mu_i}{\rho}}, \quad \lambda_{3+i} = -\lambda_i \tag{47}$$

are the real roots of the characteristic equation

$$\det \mathbf{N}(\mathbf{n}, \lambda) = 0, \tag{48}$$

where $\mathbf{N}$ is the matrix

$$\mathbf{N}(\mathbf{n}, \lambda) = \rho\lambda^2\mathbf{1} - \mathbf{B}(\mathbf{n}). \tag{49}$$

As we show in Appendix A, the solution to eqn (43) with conditions (44) is given by

$$\mathbf{w}(\mathbf{n}, \omega, t) = \frac{1}{8\pi^2}\sum_{\lambda_k}\delta'(\omega - \lambda_k t)\operatorname*{res}_{\lambda=\lambda_k}\{\mathbf{N}^{-1}(\mathbf{n}, \lambda)\}\mathbf{F}(\mathbf{n}), \tag{50}$$

where it is important to emphasize that the summation is effected over distinct roots λ_k only and res$\{\cdot\}$ involves the residues of $\mathbf{N}^{-1}$. The use of this expression in eqn (41) yields

$$\mathbf{v}(\mathbf{n}, \omega, t) = \frac{H(t)}{8\pi^2}\sum_{\lambda_k}\delta'(\omega - \lambda_k t)\operatorname*{res}_{\lambda=\lambda_k}\{\mathbf{N}^{-1}(\mathbf{n}, \lambda)\}\mathbf{F}(\mathbf{n}), \tag{51}$$

which in turn allows us to find the function ψ after integrating eqn (38), i.e.

$$\psi(\mathbf{n}, \omega, t) = \frac{1}{\alpha(\mathbf{n})}\left[\mathbf{v}(\mathbf{n}, \omega, t)\cdot\mathbf{a}(\mathbf{n}) - \frac{\delta(t)\delta(\omega)}{8\pi^2}\delta_{4M}\right]. \tag{52}$$

Expressions (51) and (52) determine the components of a fixed column of $\mathbf{V}$ and, therefore, of the matrix $\mathbf{U}$ through eqn (35). The final expressions for the components of the latter can be simplified by noting that the delta function satisfies the relation

$$\delta'(\omega - \lambda_k t) = -\frac{|\lambda_k|}{\lambda_k^3}\dot{\delta}(\Lambda), \quad \Lambda = t - \mathbf{x}\cdot\frac{\mathbf{n}}{\lambda_k} \tag{53}$$

and that the integral over the unit sphere of the second term in eqn (52) can be calculated explicitly yielding†

†Equation (54) is a consequence of a result on fundamental functions for homogeneous second order elliptic differential operators (Gelfand and Shilov, 1964).

$$\frac{1}{2\pi}\int_{|\mathbf{n}|=1}\frac{\delta(\omega)}{\alpha(\mathbf{n})}\,\mathrm{d}\Omega(\mathbf{n}) = \frac{1}{\sqrt{\alpha_1(\mathbf{x})}}, \quad \alpha_1(\mathbf{x}) = \epsilon^c_{ik}x_ix_k, \tag{54}$$

where ϵ^c_{ik} are the cofactors of the dielectric tensor ϵ_{ik}. In addition, the eigenvalues λ_k have the following properties:

$$\lambda_k(\mathbf{n}) = \lambda_k(-\mathbf{n}) \tag{55}$$

and

$$\operatorname*{res}_{\lambda=\lambda_k}\{\mathbf{N}^{-1}(\mathbf{n},\lambda)\} = \operatorname*{res}_{\lambda=\lambda_k}\{\mathbf{N}^{-1}(-\mathbf{n},\lambda)\} = -\operatorname*{res}_{\lambda=-\lambda_k}\{\mathbf{N}^{-1}(\mathbf{n},\lambda)\}. \tag{56}$$

As a result of these symmetries, the sums appearing in eqn (51), which were made over distinct roots, reduce to the sum over only positive distinct roots. Thus, the use of eqns (51)–(56) in (35) yields

$$U_{iM}(\mathbf{x},t) = -\frac{H(t)}{4\pi^2}\frac{\partial}{\partial t}\int_{|\mathbf{n}|=1}\sum_{\lambda_k>0}\delta(\Lambda)\operatorname*{res}_{\lambda=\lambda_k}\{\lambda^{-2}N^{-1}_{ij}(\mathbf{n},\lambda)\}F_{jM}(\mathbf{n})\,\mathrm{d}\Omega(\mathbf{n}) \tag{57}$$

$$U_{4M}(\mathbf{x},t) = -\frac{H(t)}{4\pi^2}\frac{\partial}{\partial t}\int_{|\mathbf{n}|=1}\sum_{\lambda_k>0}\delta(\Lambda)\operatorname*{res}_{\lambda=\lambda_k}\{\lambda^{-2}N^{-1}_{ij}(\mathbf{n},\lambda)\}F_{jM}(\mathbf{n})\frac{a_i(\mathbf{n})}{\alpha(\mathbf{n})}\mathrm{d}\Omega(\mathbf{n})$$

$$-\frac{\delta(t)\delta_{4M}}{4\pi\sqrt{\alpha_1(\mathbf{x})}}, \tag{58}$$

which are the dynamic fundamental solutions (or Green's functions) for piezoelectricity written in component form as integrals over the unit sphere. These formulas, together with properties (55) and (56), give an additional symmetry property for **U**, namely

$$\mathbf{U}(\mathbf{x}-\xi,t) = \mathbf{U}(\xi-\mathbf{x},t). \tag{59}$$

Finally, by means of eqns (27) and (59) we can write

$$\mathbf{U}^{\mathrm{T}}(\mathbf{x}-\xi,t) = \mathbf{U}(\mathbf{x}-\xi,t). \tag{60}$$

5. ONE-DIMENSIONAL INTEGRAL REPRESENTATIONS

A substantial economy of numerical calculations can be obtained if the dynamic fundamental solutions are evaluated by means of line integrals rather than surface integrals. In this section we show that such a one-dimensional integral representation is possible by using as an intermediate step the concept of the slowness surface (Burridge, 1967; Musgrave, 1970). The slowness surface S is created by means of a vector **s** whose direction is that of the vector **n** and whose magnitude is equal to the inverse of the speed of a plane wave moving in the **n**-direction. That is, if $\mathbf{s}\in S$, then

$$\mathbf{s} = \frac{\mathbf{n}}{\lambda_j(\mathbf{n})}. \tag{61}$$

Since from eqn (48)

$$\det \mathbf{N}(\mathbf{n}, \lambda) = \lambda^6 \det \mathbf{N}\left(\frac{\mathbf{n}}{\lambda}, 1\right), \tag{62}$$

the slowness surface can be represented by

$$Q(\mathbf{s}) = \det \mathbf{N}(\mathbf{s}, 1) = 0, \tag{63}$$

which consists of three smooth and closed slowness sheets S_1, S_2 and S_3. We note that for elastic isotropic materials two of these sheets coincide (i.e. they are identical). On the other hand, when the material is strongly anisotropic, as is the case of piezoelectrics, these sheets are all distinct and, therefore, do not overlap except at certain points defining an intersection line l_0, that mathematically can be expressed as

$$l_0 = \{Q(\mathbf{s}) = 0, \quad \nabla Q(\mathbf{s}) = 0\}. \tag{64}$$

For a graphical representation of eqns (63) and (64) from a purely elastic point of view, the reader is referred to Musgrave (1970), where slowness surfaces are shown for several materials.

Next, let us introduce an intersection line determined by the slowness surface S and the plane $\mathbf{x} \cdot \mathrm{s} = t$ moving in the direction $\mathbf{x}$ with unit speed, i.e.

$$l(\mathbf{x}, t) = \{Q(\mathbf{s}) = 0\} \cap \{\mathbf{x} \cdot \mathbf{s} = t\}. \tag{65}$$

The passage from the unit sphere to the slowness surface requires a relationship between the corresponding elements of area. This relation is given by

$$\mathrm{d}\Omega(\mathbf{n}) = \frac{|\mathbf{s} \cdot \nabla Q|}{|\mathbf{s}|^3 |\nabla Q|} \mathrm{d}S(\mathbf{s}), \tag{66}$$

where $\mathrm{d}S$ is the element of area of the slowness surface. Furthermore, if $l(\mathbf{x}, t)$ and l_0 intersect at most at a finite number of points, eqns (57) and (58) reduce to (see Appendix B)

$$U_{iM}(\mathbf{x}, t) = \frac{H(t)}{4\pi^2} \frac{\partial}{\partial t} \int_{Q=0} \frac{\operatorname{sgn}(\mathbf{s} \cdot \nabla Q)}{|\nabla Q|} P_{ij}(\mathbf{s}) F_{jM}(\mathbf{s}) \delta(t - \mathbf{x} \cdot \mathbf{s}) \, \mathrm{d}S(\mathbf{s}) \tag{67}$$

$$U_{4M}(\mathbf{x}, t) = \frac{H(t)}{4\pi^2} \frac{\partial}{\partial t} \int_{Q=0} \frac{\operatorname{sgn}(\mathbf{s} \cdot \nabla Q)}{\alpha(\mathbf{s}) |\nabla Q|} P_{ij}(\mathbf{s}) F_{jM}(\mathbf{s}) a_i(\mathbf{s}) \delta(t - \mathbf{x} \cdot \mathbf{s}) \, \mathrm{d}S(\mathbf{s})$$

$$- \frac{\delta(t) \delta_{4M}}{4\pi \sqrt{\alpha_1(\mathbf{x})}}, \tag{68}$$

where $\operatorname{sgn}(\cdot)$ is the signum function and

$$\mathbf{P}(\mathbf{s}) = Q(\mathbf{s}) \mathbf{N}^{-1}(\mathbf{s}, 1)$$

is the transpose of the matrix of cofactors of $\mathbf{N}(\mathbf{s}, 1)$ or so-called adjoint matrix of $\mathbf{N}$.

It must be emphasized that expressions (67) and (68) are valid for all possible values of time except for a time t at which the plane $\mathbf{x} \cdot \mathbf{s} = t$ includes part of the line l_0, in which case the integrands become indeterminate. A typical example of such an event is furnished by the case of a transversely isotropic material when its axis of symmetry coincides with $\mathbf{x}$. Even under this condition there are a limited number of times for which the lines $l(\mathbf{x}, t)$

and l_0 can overlap. For these isolated values of time, say t_0, we can still calculate eqns (67) and (68) as the limiting cases of integrals evaluated in the neighborhood of t_0.

Expressions (67) and (68) can now be reduced to one-dimensional integrals along the line $l(\mathbf{x}, t)$. For the points $\mathbf{s}$ on this line we have

$$\mathrm{d}S(\mathbf{s}) = \mathrm{d}l(\mathbf{s})\,\mathrm{d}m(\mathbf{s}),$$

where $\mathrm{d}l$ is the element of the line $l(\mathbf{x}, t)$ and $\mathrm{d}m$ is the element of the normal $\mathbf{m}$ to $l(\mathbf{x}, t)$ that is tangent to the surface $Q = 0$. Moreover, since $\mathbf{m}$ is a linear combination of the vectors $\mathbf{x}$ and ∇Q, and $\mathbf{m}\cdot\nabla Q = 0$, we can set

$$\mathbf{m} = c[|\nabla Q|^2\mathbf{x} - (\mathbf{x}\cdot\nabla Q)\nabla Q]$$

with c a constant, and from where we deduce that

$$\frac{\partial(\mathbf{x}\cdot\mathbf{s})}{\partial m} = \frac{\sqrt{|\mathbf{x}|^2|\nabla Q|^2 - (\mathbf{x}\cdot\nabla Q)^2}}{|\nabla Q|}.$$

Consequently, after some mathematical manipulations, eqn (67) reduces to

$$U_{iM}(\mathbf{x}, t) = \frac{H(t)}{4\pi^2}\frac{\partial}{\partial t}\int_{l(\mathbf{x},t)} \frac{\operatorname{sgn}(\mathbf{s}\cdot\nabla Q)P_{ij}(\mathbf{s})F_{jM}(\mathbf{s})}{\sqrt{|\mathbf{x}|^2|\nabla Q|^2 - (\mathbf{x}\cdot\nabla Q)^2}}\,\mathrm{d}l(\mathbf{s}), \tag{69}$$

while eqn (68) becomes

$$U_{4M}(\mathbf{x}, t) = \frac{H(t)}{4\pi^2}\frac{\partial}{\partial t}\int_{l(\mathbf{x},t)} \frac{\operatorname{sgn}(\mathbf{s}\cdot\nabla Q)P_{ij}(\mathbf{s})F_{jM}(\mathbf{s})a_i(\mathbf{s})}{\alpha(\mathbf{s})\sqrt{|\mathbf{x}|^2|\nabla Q|^2 - (\mathbf{x}\cdot\nabla Q)^2}}\,\mathrm{d}l(\mathbf{s}) - \frac{\delta(t)\delta_{4M}}{4\pi\sqrt{\alpha_1(\mathbf{x})}}. \tag{70}$$

Notice that when the vector $\nabla Q(\mathbf{s})$ (for $\mathbf{s}\in l(\mathbf{x}, t)$) is parallel to $\mathbf{x}$, the denominators in the integrands of eqns (69) and (70) vanish. Moreover, if t is such a value of time that $l(\mathbf{x}, t)$ contains the points $\mathbf{s}$ for which $\nabla Q(\mathbf{s})$ is parallel to $\mathbf{x}$, we should expect integrals (69) and (70) to have discontinuities when passing through this time.

To conclude this section we must emphasize that the solutions (69) and (70) have the following properties:

$$\mathbf{U}(\mathbf{x}, t) = 0 \quad \text{if } t < \frac{|\mathbf{x}|}{c_{\max}}$$

$$\mathbf{U}(\mathbf{x}, t) = 0 \quad \text{if } t > \frac{|\mathbf{x}|}{c_{\min}},$$

where $c_{\max}$ and $c_{\min}$ are the maximum and minimum wave speeds among all possible directions of propagation. If $|\mathbf{x}| = c_{\max}t$, we say that the wave front is at $\mathbf{x}$, while if $|\mathbf{x}| = c_{\min}t$, we say that at $\mathbf{x}$ we have the trailing edge of the wave. In passing, we note that the presence of the trailing edge of the wave plays an important role in the computational implementation of the fundamental solution for the BEM (Ugodchikov and Khutoryansky, 1986).

6. SINGULAR SOLUTIONS

Here we turn our attention to the matrix $\mathbf{T}$. We start by noticing that each column of $\mathbf{T}(\xi, \mathbf{x}, t)$, as a function of ξ and t, is a solution to eqn (5) with $\mathbf{f}(\xi, t) = \mathbf{0}$ and $q(\xi, \mathbf{t}) = 0$ when $\xi \neq \mathbf{x}$. This solution is named the singular solution because its singularities are one order higher than the singularities corresponding to the fundamental solution $\mathbf{U}$. In the present section we show that the components of $\mathbf{T}$ are derived from $\mathbf{U}$ using the constitutive equations and, furthermore, that they can also be evaluated in terms of one-dimensional integrals over the line $l(\mathbf{x}, t)$. For that purpose we introduce the tensor $\mathbf{S}$ such that

$$T_{MN}(\xi, \mathbf{x}, t) = S_{NjM}(\mathbf{x}-\xi, t)v_j(\mathbf{x}), \quad \mathbf{x} \in \partial\mathscr{B}. \tag{71}$$

In the framework of the paper's notation, S_{NjM} denotes the stress (if $N = 1, 2, 3$) or normal component of $\mathbf{D}$ (if $N = 4$) due to the concentrated source of mechanical ($M = 1, 2, 3$) or electrical ($M = 4$) nature. In terms of $\mathbf{S}$ and $\mathbf{U}$, the constitutive relations become

$$S_{ijM}(\mathbf{x}-\xi, t) = C_{ijkl}\frac{\partial}{\partial x_l}U_{kM}(\mathbf{x}-\xi, t)+e_{lij}\frac{\partial}{\partial x_l}U_{4M}(\mathbf{x}-\xi, t)$$

$$S_{4jM}(\mathbf{x}-\xi, t) = e_{jkl}\frac{\partial}{\partial x_l}U_{kM}(\mathbf{x}-\xi, t)-\epsilon_{jl}\frac{\partial}{\partial x_l}U_{4M}(\mathbf{x}-\xi, t). \tag{72}$$

Therefore, it is clear that the calculation of $\mathbf{T}$ reduces to calculating the derivatives of the fundamental solution $\mathbf{U}$. To obtain a one-dimensional integral formula for these derivatives, we make use of eqns (67) and (68), which yields

$$\frac{\partial}{\partial x_k}U_{iM}(\mathbf{x}, t) = -\frac{H(t)}{4\pi^2}\frac{\partial^2}{\partial t^2}\int_{Q=0}\frac{s_k\,\mathrm{sgn}\,(\mathbf{s}\cdot\nabla Q)}{|\nabla Q|} \times P_{ij}(\mathbf{s})F_{jM}(\mathbf{s})\delta(t-\mathbf{x}\cdot\mathbf{s})\,\mathrm{d}S(\mathbf{s}) \tag{73}$$

$$\frac{\partial}{\partial x_k}U_{4M}(\mathbf{x}, t) = -\frac{H(t)}{4\pi^2}\frac{\partial^2}{\partial t^2}\int_{Q=0}\frac{s_k\,\mathrm{sgn}\,(\mathbf{s}\cdot\nabla Q)}{\alpha(\mathbf{s})|\nabla Q|}P_{ij}(\mathbf{s})F_{jM}(\mathbf{s})a_i(\mathbf{s}) \times\delta(t-\mathbf{x}\cdot\mathbf{s})\,\mathrm{d}S(\mathbf{s})+\frac{\delta(t)\delta_{4M}\epsilon^C_{kj}x_j}{4\pi[\alpha_1(\mathbf{x})]^{3/2}}. \tag{74}$$

Following the steps of the previous section we can rewrite these expressions in one-dimensional form, namely

$$\frac{\partial}{\partial x_k}U_{iM}(\mathbf{x},t) = -\frac{H(t)}{4\pi^2}\frac{\partial^2}{\partial t^2}\int_{l(\mathbf{x},t)}\frac{\mathrm{sgn}\,(\mathbf{s}\cdot\nabla Q)s_kP_{ij}(\mathbf{s})F_{jM}(\mathbf{s})}{\sqrt{|\mathbf{x}|^2|\nabla Q|^2-(\mathbf{x}\cdot\nabla Q)^2}}\mathrm{d}l(\mathbf{s}) \tag{75}$$

$$\frac{\partial}{\partial x_k}U_{4M}(\mathbf{x}, t) = -\frac{H(t)}{4\pi^2}\frac{\partial^2}{\partial t^2}\int_{l(\mathbf{x},t)}\frac{\mathrm{sgn}\,(\mathbf{s}\cdot\nabla Q)s_kP_{ij}(\mathbf{s})F_{jM}(\mathbf{s})a_i(\mathbf{s})}{\alpha(\mathbf{s})\sqrt{|\mathbf{x}|^2|\nabla Q|^2-(\mathbf{x}\cdot\nabla Q)^2}}\mathrm{d}l(\mathbf{s}) +\frac{\delta(t)\delta_{4M}\epsilon^C_{kj}x_j}{4\pi[\alpha_1(\mathbf{x})]^{3/2}}. \tag{76}$$

Knowledge of the derivatives of $\mathbf{U}$ determines $\mathbf{S}$ and, therefore, the singular solutions $\mathbf{T}$ through eqn (71). It is interesting to note that in terms of a numerical implementation the first and second derivatives with respect to time appearing in eqns (69), (70) for $\mathbf{U}$ and in eqns (75), (76) for $\mathbf{T}$ do not need to be calculated. This is so because in eqn (28) the integrals in time become the convolution of the functions involved in each integrand, and since $\dot{f}*g = f*\dot{g}$, rather than taking the derivatives of $\mathbf{U}$ and $\mathbf{T}$ we only need to differentiate the vectors $\mathbf{p}$, $\mathbf{d}$ and $\mathbf{b}$ with respect to time.

7. THE SINGULARITIES OF THE REPRESENTATION FORMULA

We recall that the three main preliminary components of a BEM are: (i) representation formulas, (ii) fundamental solutions and (iii) boundary integral equations. The first two items have been addressed in the previous sections. The third item depends strongly upon the behavior of the integrands involved in eqn (28) at the boundary of the body. Consequently, knowledge of the types of singularities involved in such integrands becomes an issue of fundamental importance.

We note that the representation formula (28) has two boundary integrals: the so-called potentials of single and double layer. The integrand of the first potential (which involves $\mathbf{U}$) has a singularity of order $|\mathbf{x}-\xi|^{-1}$, while the integrand of the second potential (which involves $\mathbf{T}$) has a singularity of order $|\mathbf{x}-\xi|^{-2}$. Thus, it follows that the first integral does not have jumps when the boundary is approached from within the body. On the other hand, the leading singularity of the second potential is of exactly the same order as the singularity arising in static problems. For this reason it is useful to recover from the dynamic equations the corresponding fundamental and singular solutions of static piezoelectricity. These solutions are obtained by simply integrating eqns (57) and (58) with respect to time:

$$U^s_{iM}(\mathbf{x}) = \frac{1}{8\pi^2|\mathbf{x}|}\int_{l_1(\mathbf{x})} B^{-1}_{ij}(\mathbf{n})F_{jM}(\mathbf{n})\,\mathrm{d}l(\mathbf{n}) \tag{77}$$

$$U^s_{4M}(\mathbf{x}) = \frac{1}{8\pi^2|\mathbf{x}|}\int_{l_1(\mathbf{x})} B^{-1}_{ij}(\mathbf{n})F_{jM}(\mathbf{n})\frac{a_i(\mathbf{n})}{\alpha(\mathbf{n})}\,\mathrm{d}l(\mathbf{n}) - \frac{\delta_{4M}}{4\pi\sqrt{\alpha_1(\mathbf{x})}}, \tag{78}$$

where the superscript s stands for "static" and $l_1(\mathbf{x})$ is the intersection of the unit sphere $|\mathbf{n}| = 1$ with the plane $\mathbf{x}\cdot\mathbf{n} = 0$. It is clear that expressions (77) and (78) have singularities of order $|\mathbf{x}|^{-1}$. Moreover, the singular solutions $\mathbf{T}^s(\xi, \mathbf{x})$ for static piezoelectricity can be obtained by first differentiating eqns (77) and (78), which gives

$$\frac{\partial U^s_{iM}(\mathbf{x})}{\partial x_k} = -\frac{x_j}{8\pi^2|\mathbf{x}|^3}\int_{l_1(\mathbf{x})}\frac{\partial}{\partial n_j}[n_k B^{-1}_{il}(\mathbf{n})F_{lM}(\mathbf{n})]\,\mathrm{d}l(\mathbf{n}) \tag{79}$$

$$\frac{\partial U^s_{4M}(\mathbf{x})}{\partial x_k} = -\frac{x_j}{8\pi^2|\mathbf{x}|^3}\int_{l_1(\mathbf{x})}\frac{\partial}{\partial n_j}\left[n_k B^{-1}_{il}(\mathbf{n})F_{lM}(\mathbf{n})\frac{a_i(\mathbf{n})}{\alpha(\mathbf{n})}\right]\mathrm{d}l(\mathbf{n}) + \frac{\delta_{4M}\epsilon^{\varepsilon}_{kj}x_j}{4\pi[\alpha_1(\mathbf{x})]^{3/2}}. \tag{80}$$

Second, these results are substituted into eqn (72), which in turn are introduced in eqn (71), from where we deduce that the singularities of $\mathbf{T}^s(\xi, \mathbf{x})$ are of order $|\mathbf{x}-\xi|^{-2}$. What is crucial in this analysis is the fact that the expression

$$\mathbf{T}(\xi, \mathbf{x}, t)*\mathbf{d}(\mathbf{x}, t) - \mathbf{T}^s(\xi, \mathbf{x})\mathbf{d}(\xi, t) \tag{81}$$

has only weak singularity of type $|\mathbf{x}-\xi|^{-1}$. As a consequence, the boundary integral of eqn (81) does not have jumps when the boundary is approached from within the body. We are now ready to discuss the third important aspect of the BEM.

8. THE BOUNDARY INTEGRAL EQUATION

This is obtained by means of the following procedure: a constant vector $\mathbf{d}$ is substituted into eqn (28), and since in this case the vectors $\mathbf{p}$ and $\mathbf{b}$ are zero, we have that

$$\int_{\partial\mathscr{B}} \mathbf{T}^s(\xi, \mathbf{x})\,\mathrm{d}A(\mathbf{x}) = -\mathbf{I}, \quad \mathbf{x}\in\mathscr{B},$$

which is a generalized form of Gauss' integral. Multiplying both members by $\mathbf{d}(\xi, t)$ gives

$$\int_{\partial \mathscr{B}} \mathbf{T}^{s}(\xi, \mathbf{x})\mathbf{d}(\xi, t)\,\mathrm{d}A(\mathbf{x}) = -\mathbf{d}(\xi, t), \quad \mathbf{x} \in \mathscr{B},$$

from where the representation formula (28) can be rewritten as

$$\int_{\partial \mathscr{B}} \left[\int_0^t \mathbf{T}(\xi, \mathbf{x}, t-\tau)\mathbf{d}(\mathbf{x}, \tau)\,\mathrm{d}\tau - \mathbf{T}^{s}(\xi, \mathbf{x})\mathbf{d}(\mathbf{x}, t) \right] \mathrm{d}A(\mathbf{x}) - \int_{\partial \mathscr{B}} \int_0^t \mathbf{U}(\xi - \mathbf{x}, t-\tau)\mathbf{p}(\mathbf{x}, \tau)\,\mathrm{d}\tau\,\mathrm{d}A(\mathbf{x})$$

$$= \int_{\mathscr{B}} \int_0^t \mathbf{U}(\xi - \mathbf{x}, t-\tau)\mathbf{b}(\mathbf{x}, \tau)\,\mathrm{d}\tau\,\mathrm{d}V(\mathbf{x}), \quad \xi \in \mathscr{B}. \quad (82)$$

The importance of this formula resides in the fact that now the integrands of both boundary integrals have a weak singularity of order $|\mathbf{x} - \xi|^{-1}$. Consequently, the limit form of eqn (82), as the observation point ξ tends to the boundary $\partial \mathscr{B}$, coincides with (82) on the boundary (i.e. $\xi \in \partial \mathscr{B}$), where it holds not only on smooth points of $\partial \mathscr{B}$ but also on points or lines where the normal does not exist. Equation (82) on $\partial \mathscr{B}$ is the regularized form of the boundary integral equation for dynamic piezoelectricity.

Finally, to obtain the complete system of boundary integral equations containing the unknown variables one must consider, in addition to eqn (82), the boundary integral equations representing the fields of the surroundings (for example, the environment or other deformable dielectrics or conductors) of the body. In this respect we want to bring to the reader's attention eqn (6d). We recall that at the dielectric–dielectric interface there is no surface charge, i.e. $\varpi = 0$. In such a case, both dielectric media (which in general may have different material characteristics) will have the representation (82) together with the condition of continuity of the normal component of the electric displacement. The same is true for two electrically conducting bodies. However, at the boundary between a dielectric and a conductor the surface charge is not zero and a jump of the normal component of the electric displacement takes place. In this case, in addition to eqn (82) for the dielectric body we need a boundary integral equation for the elastic conductor (Ugodchikov and Khutoryansky, 1986). Finally, we can take a step further to note that if the piezoelectric body is surrounded by air or vacuum one can use, to a good approximation, the condition $\mathbf{D} \cdot \boldsymbol{\nu} = 0$, where $\mathbf{D}$ corresponds to the electric displacement within the body. This approximation stems from the fact that the dielectric permittivity of a piezoelectric material is usually three orders of magnitude higher than the permittivity of air or vacuum.

9. CONCLUSIONS

We have been concerned with a three-dimensional electroelastic analysis of homogeneous, anisotropic dielectrics. In contrast with previous formulations, transient effects have been taken into consideration. This paper provides the mathematical foundations for the development of the BEM and its application to problems involving piezoelectric materials. Consequently, the article has been devoted to three main subjects. (a) The derivation of a representation formula by means of a generalization of the reciprocal theorem of elastodynamics. This formula includes fundamental and singular solutions for the infinite piezoelectric medium as kernels of two boundary and one volume integrals. (b) The derivation of fundamental and singular solutions to the transient equations of piezoelectricity. These solutions have been first obtained through the plane wave transform and represented by means of integrals over the unit sphere. Furthermore, we have shown that by introducing the concept of the slowness surface, alternative and simpler integral representations can be deduced. Of particular interest has been the description of the fundamental and singular solutions over one-dimensional integrals which are both simple in form and suitable for numerical computations. (c) An analysis of the behavior of the representation formula when the boundary is approached from within the body. Here we have shown that by using the corresponding static singular solutions, a regularized form of the representation formula can be obtained which is valid not only at the boundary but also at points where the normal does not exist.

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APPENDIX A

Let $g(z)$ be an analytic function of the complex variable z with a cut on the positive imaginary axis. As a consequence, $g(\omega-\lambda t)$ is analytic in the complex plane λ with a cut c_1 along an axis parallel to the imaginary axis whose origin is at the point $\lambda_0 = \omega/t$ and its direction is dictated by the sign of t. Furthermore, let $\mathscr{C}$ be a simple closed curve in the complex plane λ containing in its interior the roots of the characteristic equation (48) but excluding the point λ_0. The curve $\mathscr{C}$ exists if

$$\omega \neq t\lambda_k(\mathbf{n}), \quad k = 1,\ldots,6, \tag{A1}$$

in which case $\mathbf{w}$ is of the form

$$\mathbf{w}(\mathbf{n},\omega,t) = \mathscr{R}\left\{\frac{1}{2\pi i}\oint_{\mathscr{C}} g(\omega-\lambda t)\mathbf{N}^{-1}(\mathbf{n},\lambda)\mathbf{F}(\mathbf{n})\,\mathrm{d}\lambda\right\},$$

where $\mathscr{R}\{\cdot\}$ denotes the real part of the function enclosed within braces. Applying the operator $\mathbf{N}(\nabla,\partial_t)$ to this expression yields

$$\rho\ddot{\mathbf{w}} - \mathbf{B}\mathbf{w}'' = \mathscr{R}\left\{\frac{1}{2\pi i}\oint_{\mathscr{C}} g''(\omega-\lambda t)\,\mathrm{d}\lambda\right\}\mathbf{F}(\mathbf{n}),$$

whose right-hand side vanishes because of Cauchy's theorem. Hence $\mathbf{w}$ obeys eqn (43) if (A1) holds true and can be written (due to the residue theorem) as

$$\mathbf{w}(\mathbf{n},\omega,t) = \sum_{\lambda_k} \mathscr{R}\{g(\omega-\lambda_k t)\} \operatorname*{res}_{\lambda=\lambda_k}\{\mathbf{N}^{-1}(\mathbf{n},\lambda)\}\mathbf{F}(\mathbf{n}). \tag{A2}$$

In turn, the initial conditions (44) become

$$\mathbf{w}(\mathbf{n},\omega,0) = \mathscr{R}\{g(\omega)\}\sum_{\lambda_k} \operatorname*{res}_{\lambda=\lambda_k}\{\mathbf{N}^{-1}(\mathbf{n},\lambda)\}\mathbf{F}(\mathbf{n})$$

$$\dot{\mathbf{w}}(\mathbf{n},\omega,0) = -\mathscr{R}\{g'(\omega)\}\sum_{\lambda_k} \operatorname*{res}_{\lambda=\lambda_k}\{\lambda\mathbf{N}^{-1}(\mathbf{n},\lambda)\}\mathbf{F}(\mathbf{n}).$$

Since $\mathbf{N}(\mathbf{n}, \lambda)$ and $\lambda\mathbf{N}^{-1}(\mathbf{n}, \lambda)$, as functions of the complex variable λ, are analytic in the entire complex plane except for the poles given by eqn (47), by the residue theorem we have

$$\sum_{\lambda_k} \operatorname*{res}_{\lambda=\lambda_k} \{\mathbf{N}^{-1}(\mathbf{n}, \lambda)\} = -\operatorname*{res}_{\lambda=\infty} \{\mathbf{N}^{-1}(\mathbf{n}, \lambda)\} = 0$$

$$\sum_{\lambda_k} \operatorname*{res}_{\lambda=\lambda_k} \{\lambda\mathbf{N}^{-1}(\mathbf{n}, \lambda)\} = -\operatorname*{res}_{\lambda=\infty} \{\lambda\mathbf{N}^{-1}(\mathbf{n}, \lambda)\} = \frac{1}{\rho}\mathbf{1},$$

which produces

$$\mathbf{w}(\mathbf{n}, \omega, 0) = \mathbf{0}$$
$$\dot{\mathbf{w}}(\mathbf{n}, \omega, 0) = -\frac{1}{\rho}\mathscr{R}\{g'(\omega)\}\mathbf{F}(\mathbf{n}). \tag{A3}$$

Comparing eqn (A3) with (44) we obtain

$$\mathscr{R}\{g(\omega)\} = -\frac{1}{8\pi^2}\delta'(\omega),$$

which holds true if we extend the real function g to the whole complex plane. To that purpose we choose (John, 1955)

$$g(z) = g_0''(z), \quad g_0(z) = \frac{1}{16\pi^2}\left[1 + \frac{1}{\pi i}\ln z^2\right]$$

because

$$\mathscr{R}\{g_0(z)\} = \frac{1}{16\pi^2}\operatorname{sgn}[\mathscr{R}\{z\}].$$

As a consequence, eqn (A2) becomes (50), i.e.

$$\mathbf{w}(\mathbf{n}, \omega, t) = -\frac{1}{8\pi^2}\sum_{k}\delta'(\omega - \lambda_k t)\operatorname*{res}_{\lambda=\lambda_k}\{\mathbf{N}^{-1}(\mathbf{n}, \lambda)\}\mathbf{F}(\mathbf{n}).$$

APPENDIX B

Let us consider the inverse of $\mathbf{N}$, i.e.

$$\mathbf{N}^{-1} = \frac{\mathbf{N}^a}{\det \mathbf{N}},$$

where $\mathbf{N}^a$ is the adjoint matrix of $\mathbf{N}$. The denominator of this expression has simple poles provided the roots of the determinant are distinct. Thus, by virtue of the residue theorem we have

$$\operatorname*{res}_{\lambda=\lambda_k}\{\mathbf{N}^{-1}(\mathbf{n}, \lambda)\} = \left[\frac{\mathbf{N}^a(\mathbf{n}, \lambda)}{\frac{\partial}{\partial\lambda}\det \mathbf{N}(\mathbf{n}, \lambda)}\right]_{\lambda=\lambda_k}.$$

Furthermore, since

$$\det \mathbf{N}(\mathbf{n}, \lambda_k) = 0$$

and $\det \mathbf{N}(\mathbf{n}, \lambda)$ is homogeneous in both arguments, it follows that

$$\left[\lambda \frac{\partial}{\partial \lambda} \det \mathbf{N}(\mathbf{n}, \lambda) + n_i \frac{\partial}{\partial n_i} \det \mathbf{N}(\mathbf{n}, \lambda)\right]_{\lambda = \lambda_k} = 0,$$

which allows us to write

$$\operatorname*{res}_{\lambda = \lambda_k} \{\mathbf{N}^{-1}(\mathbf{n}, \lambda)\} = \frac{-\lambda_k \mathbf{N}^a(\mathbf{n}, \lambda_k)}{\mathbf{n} \cdot \nabla[\det \mathbf{N}(\mathbf{n}, \lambda_k)]}.$$

Therefore, after some mathematical manipulations we obtain

$$-\frac{1}{\lambda^2} \operatorname*{res}_{\lambda = \lambda_k} \{\mathbf{N}^{-1}(\mathbf{n}, \lambda)\} = \frac{\mathbf{N}^a(\mathbf{n}, \lambda_k)}{\lambda_k \mathbf{n} \cdot \nabla[\det \mathbf{N}(\mathbf{n}, \lambda_k)]} = \frac{1}{\lambda_k^3} \frac{\mathbf{P}(\mathbf{s})}{\mathbf{s} \cdot \nabla Q},$$

where we have used the relations $\mathbf{s} = \mathbf{n}/\lambda$ and $\mathbf{P}(\mathbf{s}) = \mathbf{N}^a(\mathbf{n}/\lambda, 1)$.